Similarity and Numerical Analysis of a Singular Moving Boundary Hyperbolic Problem ⋆

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Abstract. In [SIAM Rev., **40** (1998) 616–635], we emphasized the relevance of a combination of similarity and numerical analysis for the numerical solution of moving boundary hyperbolic problems. Here we report on results obtained for one problem of the above class that is singular at the moving boundary.

1 Velocity impact to a thin rod

We pointed out in [3] the relevance of a combination of similarity and numerical analysis for the numerical solution of moving boundary hyperbolic problems. Here we report on results obtained for one problem of the above class that is singular at the moving boundary: namely, the problem of a time dependent velocity impact to a thin rod [5,1,2].

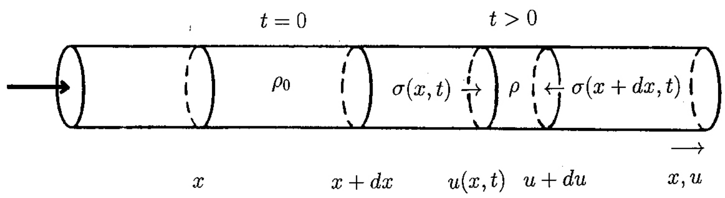

Fig. 1. Schematic representation of the one-dimensional rod.

We consider a thin rod at rest and, at the initial time, we apply a velocity impact at one end of it, for instance, by striking the rod with a hammer. We assume that the transverse movements of the rod are negligible with respect to the longitudinal ones, that is the strike has approximately the same direction of the longitudinal axis. In Fig. 1 a generic element of the rod with density $\rho_0 = const$ is denoted by the Lagrangian coordinate x and

⋆ This work is dedicated to the memory of Andrea Donato.

$x+dx$. The same element, at some time $t>0$, is shown between the Eulerian coordinates $u(x,t)$ and $u+du$ and its density is now $\rho(x,t)$. Newton's equation for this element is given by

$$\rho_0\, dx\, \frac{\partial^2 u}{\partial t^2} = \sigma(x,t) - \sigma(x+dx,t) \ , \quad \Rightarrow \quad \rho_0\, \frac{\partial^2 u}{\partial t^2} = -\frac{\partial \sigma}{\partial x} \ ,$$

where σ indicates the stress of the rod. The stress can be related to the strain

$$e(x,t) = -\frac{\partial u}{\partial x}(x,t) \ ,$$

by the constitutive law used to model the behaviour of rubbers and of some metals

$$\sigma = E_0\, e^{1/q} \ ,$$

where E_0 and q are constants with $q \neq 0$. By using the above relations we find that

$$\frac{\partial^2 u}{\partial t^2} + \frac{E_0}{\rho_0}\frac{\partial}{\partial x}\left(-\frac{\partial u}{\partial x}\right)^{1/q} = 0 \ , \tag{1}$$

is the equation governing the longitudinal movement of the rod to be considered along with the initial and boundary conditions

$$\begin{gathered} u(x,0)=0, \qquad \frac{\partial u}{\partial t}(x,0)=0, \qquad x_w(0)=0 \ , \\ \frac{\partial u}{\partial t}(0,t) = V_0\, t^{\kappa} \ , \qquad u(x_w(t),t)=0 \ , \\ \frac{\partial u}{\partial x}(x_w(t),t) = -\left[\frac{E_0}{\rho_0}\, q \left(\frac{dx_w}{dt}\right)^2\right]^{q/(1-q)} \ , \end{gathered} \tag{2}$$

where $x_w(t)$ represents the unknown moving boundary, V_0 and κ are constants with $\kappa \geq 0$. The initial conditions are related to a rod at rest, the first boundary condition is due to the time dependent velocity impact, the second is obvious and the last one was derived in [5].

2 Similarity analysis

The problem (1-2) is invariant with respect to the scaling group

$$x^* = \mu^{\gamma}\, x \ , \quad {x_w}^* = \mu^{\gamma}\, x_w \quad t^* = \mu\, t \quad u^* = \mu^{\alpha}\, u \ ,$$

provided that the conditions

$$\begin{gathered} \alpha = \kappa + 1 \ , \\ \gamma = 1 - \kappa(q-1)/(q+1) \ , \end{gathered}$$

are fulfilled. Therefore, we can define the similarity variables

$$\xi = x\, t^{-\gamma}, \quad \xi_w = x_w(t)\, t^{-\gamma} \ , \quad f(\xi) = u(x,t)\, t^{-(\kappa+1)} \ ,$$

and note that the partial derivatives transform as follows

$$\frac{\partial}{\partial t} = -\gamma\, t^{-1}\, \xi\, \frac{d}{d\xi} \ , \quad \frac{\partial}{\partial x} = t^{-\gamma}\, \frac{d}{d\xi} \ , \quad \frac{\partial \xi}{\partial t} = -\frac{\gamma\, \xi}{t} \ .$$

By taking into account the above relations we can get from (1)-(2) the ordinary differential problem

$$\begin{gathered} \left[\frac{E_0}{\rho_0}\, \frac{1}{q} \left(-\frac{df}{d\xi}\right)^{(1-q)/q} - \gamma^2\, \xi^2\right] \frac{d^2 f}{d\xi^2} - \gamma\, (\gamma - 2\kappa - 1)\, \xi\, \frac{df}{d\xi} + \\ - \kappa\, (\kappa+1) f = 0 \ , \\ f(0) = V_0/(\kappa+1) \ , \\ f(\xi_w) = 0, \qquad \frac{df}{d\xi}(\xi_w) = -\left[\frac{\rho_0}{E_0}\, q\, \gamma^2\, (\xi_w)^2\right]^{q/(1-q)} \ . \end{gathered} \tag{3}$$

Note that, at the boundary $x = 0$ we have $u(0,t) = t^{\kappa+1}\, f(0)$, so that the first boundary condition defines the value of $f(0)$. The problem (3) is a singular free boundary value problem, where ξ_w represents the unknown free boundary. The governing differential equation and the two boundary conditions at the free boundary in (3) are invariant with respect to the following scaling group

$$\xi^* = \lambda^\delta\, \xi, \qquad {\xi_w}^* = \lambda^\delta\, \xi_w, \qquad f^* = \lambda\, f \ ,$$

where $\delta = (1-q)/(1+q)$.

3 Numerical treatment and validation

For the numerical solution of (3) we apply the non-iterative transformation method [4]. A straightforward application of the method is not possible because the problem is singular at the free boundary. However, we can resort to the concept of discrete perturbation stability analysis. To this end, instead of the exact boundary conditions in (3, we consider $f^*({\xi_w}^*) = 0$ and

$$\frac{df^*}{d\xi^*}({\xi_w}^*) = -\left[\frac{\rho_0}{E_0}\, q\, \gamma^2\, ({\xi_w}^*)^2\right]^{q/(1-q)} + \epsilon^* \ , \tag{4}$$

where $|\epsilon^*| \ll 1$.

A first numerical integration of the governing equation written in the starred variables inwards on $[0, {\xi_w}^* = 1]$ with initial conditions $f^*({\xi_w}^*) = 0$ and (4), allows us to compute the values of

$$\lambda = f^*(0)/f(0) \ , \quad \xi_w = \lambda^{-\delta}\, {\xi_w}^* \ , \quad \text{and} \quad \frac{df}{d\xi}(0) = \lambda^{-2q/(q+1)}\, \frac{df^*}{d\xi^*}(0) \ .$$

In general the condition $|\epsilon^*| \ll 1$ will not imply that $|\epsilon| \ll 1$, so that we must check if the value

$$\epsilon = \lambda^{-2q/(1+q)}\, \epsilon^*$$

verifies the latter condition. The obtained value of $\frac{df}{d\xi}(0)$ can be used, along with the initial condition $f(0) = V_0/(\kappa+1)$, for a second numerical integration of the governing equation forward on $[0, \xi_w]$. Within this second integration we can verify *a posteriori* that the problem is stable with respect to variations of the boundary data, that is, we require that the computed solution verifies the conditions:

$$f(\xi_w) \approx 0\ , \quad \text{and} \quad \frac{df}{d\xi}(\xi_w) \approx -\left[\frac{\rho_0}{E_0}\, q\, \gamma^2\, (\xi_w)^2\right]^{q/(1-q)} \quad . \tag{5}$$

4 Numerical results

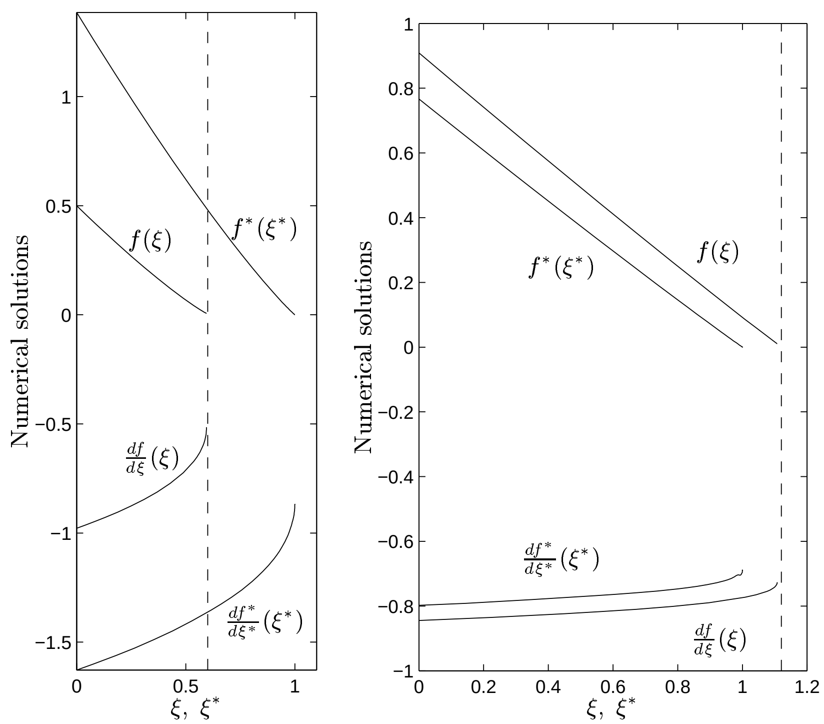

Fig. 2. Left frame: $q = 1/3, E_0/\rho_0 = 1, V_0 = 1$ and $\kappa = 1$. Right frame: $q = 1/5, E_0/\rho_0 = 1, V_0 = 1$ and $\kappa = 1/10$.

Figure 2 displays two sample computations: for the left (right) frame we used $\epsilon^* = -10^{-6}$ ($\epsilon^* = 10^{-6}$) and found $\xi_w = 0.6008$ with $|\epsilon| < 10^{-6}$

($\xi_w = 1.1205$ with $\epsilon < 2 \cdot 10^{-6}$). The location of the free boundary ξ_w is marked in Fig. 2 by a dashed line. The self-similarity of the obtained numerical solutions is evident. Moreover, both frames of Fig. 2 show the validation of the obtained numerical results according to the relations (5).

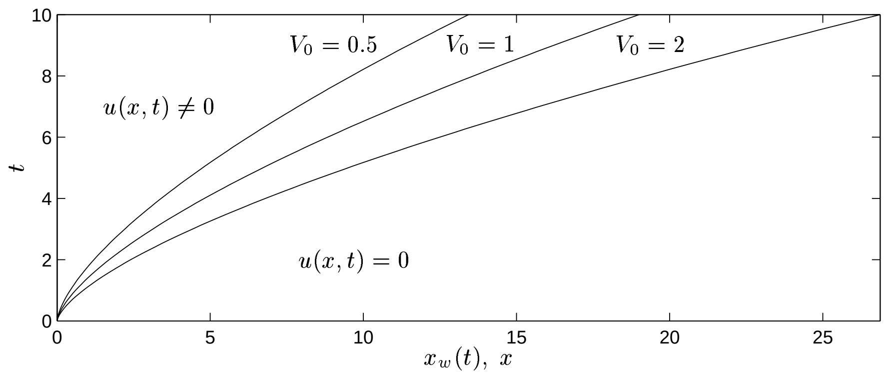

Fig. 3. Sample curves $x_w(t)$ for different values of V_0 with fixed $q = 1/3, E_0/\rho_0 = 1$ and $\kappa = 1$.

In Fig 3 we display the evolution of the moving boundary $x_w(t)$ for different time impacts within the same rod. We can easily verify that the stronger the impact (for increasing values of V_0) the faster the evolution.

From a practical point of view, our experience suggests that when dealing with singular free boundary problems we must apply a stiff solver. We used the `ODE15s` routine of the recent MATLAB ODE suite developed by Shampine and Reichelt [6] and available with the latest MATLAB releases.

References

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